where the matrix P_v has the form shown in Eq. (6). As a result, the formulas for position errors can also be applied to velocity errors, but with P_v replacing P_r. Time-rms velocity error is defined as

$$|v|_{\text{time-rms}} = \left\{ \frac{1}{T} \int_0^T E[\,|v(t)|^2\,]\,\mathrm{d}t \right\}^{1/2} \tag{18}$$

As a result, time-rms velocity error can be expressed in terms of weighting matrices

$$|v|^2_{\text{time-rms}} = \text{trace}[C_x(0)\, W_{x0}(F,P_v)] + \text{trace}[C_u W_{u0}(F,G,P_v)] \tag{19}$$

where the weighting matrices are given by Eqs. (15) and (16), but with P_r replaced by P_v, and $k=0$.

Weighting Matrices for Multimission Applications

The matrix functions F and G are mission-dependent. One may prefer to compute rms statistics over an ensemble of missions, with associated functions F_1, G_1; F_2, G_2; $\cdots$; F_m, G_m. The equivalent weighting matrices for this case are

$$W_{xk}(P) = \frac{1}{m} \sum_{i=1}^{m} W_{xk}(F_i,P) \tag{20}$$

$$W_{uk}(P) = \frac{1}{m} \sum_{i=1}^{m} W_{uk}(F_i,G_i,P) \tag{21}$$

for $P=P_r$ or P_v.

Square-Root Implementation

If R is the upper-triangular square root of C and Q is the lower-triangular square root of W,† then

$$\text{trace}(CW) = \sum S_{ij}^2 \tag{22}$$

where

$$S = RQ^{\mathrm{T}} \tag{23}$$

For $n \times n$ matrices, this computation requires in the order of $n^3/6$ multiplies, whereas computing trace (CW) requires only n^2 multiplies. However, if C is a diagonal matrix (i.e., the error budget includes no nonzero correlations), then it requires in the order of n^2 multiplies. This approach may be preferable in the case that the error budget is given in terms of rms values, because it avoids having to square them to obtain the equivalent covariance matrices.

Limitations

The model in Eq. (1) does not apply to "aided" inertial navigation, in which other navigational sensors are used for estimating and correcting the error states. It is not known whether an equivalent weighting matrix can be defined for that application.

References

[1] O'Donnell, C.F., *Inertial Navigation Analysis and Design,* McGraw-Hill, New York, 1964.

[2] Tech. Exhibit ENAC-77-1, "Characteristic for a Moderate Accuracy Inertial Navigation System (INS)," Aeronaut. Sys. Div., Air Force Systems Command, WPAFB, Ohio, March 1978.

[3] Bucy, R.S. and Joseph, P.D., *Filtering for Stochastic Processes with Applications to Guidance,* Interscience, New York, 1968, pp. 24-25.

†W may be only positive semidefinite, rather than positive definite. It is still possible to obtain a square root of W, however.

Bounds on the Solution to a Universal Kepler's Equation

John E. Prussing*
University of Illinois at Urbana-Champaign, Urbana, Ill.

Introduction

A FUNDAMENTAL problem in two-body orbital mechanics is the solution of Kepler's equation, which describes the initial-value problem relating time and position in a known orbit from specified initial position r_0 and velocity v_0 at an arbitrary epoch t_0. In this Note upper and lower bounds on the position in orbit at a specified value of time are derived for a class of universal formulations of Kepler's equation valid for all conic orbits and for the classical formulations for elliptic and hyperbolic orbits. Bounds on the solution provide efficient starting values for iterative numerical algorithms and eliminate unnecessary computations outside of the solution bounds. This is especially useful in the case of onboard computation.

Several numerical examples illustrating the use of these bounds in example problems from the textbook literature are given in Ref. 1 which is a generalization of preliminary results reported in Ref. 2. These examples demonstrate that a starting value for the iteration which is close to the desired solution is obtained using the bounds.

Analysis

Consider formulations of Kepler's equation in which a generalized position variable u, which describes the location of a body in orbit relative to its initial position at an arbitrary epoch t_0, satisfies a Sundman transformation[3] of the form

$$\mathrm{d}u/\mathrm{d}t = K/r \qquad u(t_0) = 0 \tag{1}$$

where K is a constant and r is the magnitude of the radius vector. The position variable u can then be interpreted as a regularized time variable.[4] Formulations in which the Sundman transformation is satisfied include 1) the classical Kepler's equation for an elliptical orbit in which u is the difference in eccentric anomaly $E-E_0$ and $K=\sqrt{\mu/a}$; 2) the classical hyperbolic Kepler's equation in which u is the difference in hyperbolic eccentric anomaly $H-H_0$ and $K=v_\infty$ (hyperbolic excess speed); 3) Barker's equation for a parabolic orbit; and 4) various universal formulations,[5-9] including that of Battin,[5] in which $u=x$ and $K=\sqrt{\mu}$ independent of the orbit.

Since the value of the radius r depends on the position in the orbit, Eq. (1) can be written as

$$r(u)\,\mathrm{d}u = K\,\mathrm{d}t \qquad u(t_0) = 0 \tag{2}$$

Integrating, one obtains

$$I(u) \triangleq \int_0^u r(\xi)\,\mathrm{d}\xi = K(t-t_0) \tag{3}$$

Presented as Paper 78-1405 at the AIAA/AAS Astrodynamics Conference, Palo Alto, Calif., Aug. 7-9, 1978; submitted Aug. 21, 1978; revision received May 21, 1979.

Index categories: Analytical and Numerical Methods; Earth-Orbital Trajectories; Lunar and Interplanetary Trajectories.

*Associate Professor, Department of Aeronautical and Astronautical Engineering. Associate Fellow AIAA.

where $I(u)$ is a transcendental function which must be inverted to obtain the solution for the position u at a specified time t. Obtaining the solution u can also be described as finding the root of the function $F(u)$ defined as

$$F(u) \triangleq I(u) - K(t-t_0) \tag{4}$$

Since numerical iteration is typically used to solve $F(u)=0$, it is useful to have upper and lower bounds u^+ and u^- on the solution for u. For a numerical algorithm which requires a single starting value for the iteration, such as Newton's method or successive approximations, the midpoint of the interval, $(u^- + u^+)/2$, defined by the bounds can be used. For an algorithm requiring two starting values, such as the secant method, the bounds themselves can be used.

From Eqs. (3) and (4) it is evident that

$$F'(u) = r(u) > 0 \tag{5}$$

Because of this, an interesting fringe benefit occurs with the use of a Newton iteration algorithm. The value of the derivative $F'(u)$ used to iteratively solve $F(u)=0$ converges to the value of the radius $r(u^*)$ as the value of u converges to the solution u^*. In all the formulations mentioned previously, closed-form expressions exist for $r(u)$.

Equation (5) implies that the slope of the funtion $F(u)$ is bounded by maximum and minimum values r^+ and r^- on an interval $0 \leq u \leq u^+$, where u^+ is an upper bound to be determined:

$$0 < r^- \leq F'(u) \leq r^+ \tag{6}$$

For $u \geq 0$, corresponding to $t \geq t_0$, it follows that

$$ur^- \leq I(u) \leq ur^+ \tag{7}$$

where the inequalities are reversed for $u<0$ corresponding to $t<t_0$. In the analysis which follows it is assumed that $t \geq t_0$; modifications for $t<t_0$ are straightforward. In terms of $F(u)$

$$ur^- - K(t-t_0) \leq F(u) \leq ur^+ - K(t-t_0) \tag{8}$$

By defining values

$$u^+ \triangleq K(t-t_0)/r^- \tag{9}$$

and

$$u^- \triangleq K(t-t_0)/r^+ \tag{10}$$

it can be seen from Eq. (8) that

$$F(u^-)F(u^+) \leq 0 \tag{11}$$

This proves the existence of a solution to $F(u)=0$ in the interval $u^- \leq u \leq u^+$. The fact that $F'(u)>0$ guarantees the solution is unique.

The bounds of Eqs. (9) and (10) can be interpreted in terms of the value of u on a circular orbit. From Eq. (3) it can be seen that on a circular orbit of radius r_c, $I(u_c) = r_c u_c$ and thus

$$u_c = K(t-t_0)/r_c \tag{12}$$

Bounds of Eqs. (9) and (10) are then the contemporaneous values of u on circular orbits having radii r^- and r^+, respectively.

In terms of the universal variable x, the bounds of Eqs. (9) and (10) correspond to

$$\sqrt{\mu}(t-t_0)/r^+ \leq x \leq \sqrt{\mu}(t-t_0)/r^- \tag{13}$$

The selection of the values of r^- and r^+ depends on the type of conic orbit being considered.

Ellipses

For an elliptical orbit the most conservative bounds are obtained by selecting r^- and r^+ to be the periapse and apoapse radii, r_p and r_a respectively. In terms of classical eccentric anomaly E the bounds of Eqs. (9) and (10) correspond to

$$\frac{M-M_0}{1+e} \leq E - E_0 \leq \frac{M-M_0}{1-e} \tag{14}$$

where M is the mean anomaly and e is the eccentricity. In certain applications refinements can be made which tighten the bounds:

1) Since the orbit is periodic with period P, one can always assume that $t-t_0<P$ corresponding to $u<\hat{u} \triangleq u(P+t_0)$, the value of the position variable one period after t_0. The value $\hat{u}$ can be shown to be KP/a, using Eq. (1) and the fact that the time average of $1/r$ over one period is $1/a$. Thus the upper bound can be tightened to

$$u^- \leq u \leq \min[KP/a, u^+] \tag{15}$$

For $u=E-E_0$, KP/a is simply 2π.

2) A further refinement occurs in the case $r_0^T v_0 > 0$, indicating that the initial point is on the outbound half of the ellipse. If $r(u^+) \geq r_0$, the value of the upper bound u^+ in Eq. (15) can be decreased by using $r^- = r_0 > r_p$, since $r \geq r_0$. In terms of eccentric anomaly this corresponds to an upper bound on $E-E_0$ equal to $(M-M_0)a/r_0$. It is interesting to note that if $r_0>a$ the customary starting value for the iteration, $E-E_0 = M-M_0$, is greater that this refined upper bound. This tightening of the upper bound also applies in the case of an ascent ellipse from a planet of radius R, in which case $r^- = R$.

3) In the case $r_0{}^T v_0 < 0$ and $r(u^+) \leq r_0$ the lower bound u^- can be increased by using $r^+ = r_0 < r_a$, since $r \leq r_0$.

4) If the initial time t_0 is periapse passage time and u^+ is calculated to be less than the value of u at apoapse $KP/2a$, indicating that the solution lies on the outbound half of the ellipse, the lower bound u^- can be increased by using $r^+ = r(u^+) < r_a$, since $r \leq r(u^+)$. This corresponds to a lower bound on eccentric anomaly $E^- = M/(1-e\cos E^+)$ which reduces to the conservative bound of Eq. (14) as $E^+ \to \pi$.

Hyperbolas and Parabolas

In this case the most conservative bounds are obtained by selecting $r^- = r_p$ and $r^+ = \infty$, yielding zero as the lower bound in Eq. (10). (A small positive lower bound would result if r^+ were chosen to be the radius of the sphere of influence.) In terms of hyperbolic eccentric anomaly H these conservative bounds are

$$0 \leq H - H_0 \leq v_\infty (t-t_0)/r_p \tag{16}$$

As in the elliptical case, certain refinements can be made which tighten the bounds. If $r_0^T v_0 > 0$, the upper bound can be decreased by using $r^- = r_0 > r_p$ since the radius will never decrease. If $r_0^T v_0 < 0$ and $r(u^+) \leq r_0$, the lower bound can be increased by using $r^+ = r_0 < \infty$, since $r \leq r_0$.

References

[1] Prussing, J.E., "Bounds on the Solution to a Universal Kepler's Equation," AIAA Paper 78-1405, Palo Alto, Calif., 1978.

[2] Prussing, J.E., "Bounds on the Solution to Kepler's Problem," *The Journal of the Astronautical Sciences*, Vol. XXV, April-June 1977, pp. 123-128.

[3] Sundman, K., "Memoire sur le problems de trois corps," *Acta Mathematika*, Vol. 36, 1912, p. 127.

[4]Szebehely, V., *Theory of Orbits*, Academic Press, New York, 1967.

[5]Battin, R.H., *Astronautical Guidance*, McGraw-Hill, New York, 1964, Chap. 2.

[6]Goodyear, W.H., "A General Method for the Computation of Cartesian Coordinates and Partial Derivatives of the Two-Body Problem," NASA CR-522, Sept. 1966.

[7]Stumpff, K., *Himmelsmechanik*, VES Verlag, Berlin, 1959.

[8]Herrick, S., *Astrodynamics*, Vol. I, Van Nostrand Reinhold Co., London, 1971.

[9]Pines, S., and Fang, T.C., "A Uniform Closed Solution of the Variational Equations for Optimal Trajectories during Coast," *Advanced Problems and Methods for Space Flight Optimization*, edited by F. de Veubeke, Pergammon Press, New York, 1969, p. 175.

Geometrical Interpretation of the Angles α and β in Lambert's Problem

John E. Prussing*
University of Illinois at Urbana-Champaign, Urbana, Ill.

Introduction

LAMBERT'S problem is the two-point boundary-value problem of determining the two-body orbit which connects two given position vectors in a specified flight time. In the classical formulation of the equation for the flight time on an elliptical orbit, due to Lagrange,[1,2] two angles α and β appear, whose values depend on the geometry of the terminal radii and the semimajor axis of the transfer orbit. In this Note, a simple geometrical interpretation of these angles and their analogs for hyperbolic orbits is presented which can be used to geometrically construct these angles in position space. A different interpretation of these angles by Battin[3] in velocity space requires a more complex construction, but allows one to also construct the required velocity vectors at the given terminal radii.

Definitions of α and β

The geometry of Lambert's problem is shown in Fig. 1. The points F and F^* are the focus and vacant focus, respectively, of the transfer ellipse between points P_1 and P_2, and it is assumed that $r_2 \geq r_1$ with no loss of generality. The time of flight $t_F = t_2 - t_1$ between points P_1 and P_2, separated by the chord distance c and the transfer angle θ, is given by[2]

$$\sqrt{\mu} t_F = a^{3/2} [\alpha - \beta - (\sin\alpha - \sin\beta)] \tag{1}$$

where a is the semimajor axis of the elliptical transfer orbit and the angles α and β are defined in terms of the semiperimeter of the Lambert triangle P_1FP_2 of Fig. 1, $s = (r_1 + r_2 + c)/2$ by

$$\sin(\alpha/2) = \sqrt{s/2a} \tag{2}$$

$$\sin(\beta/2) = \sqrt{(s-c)/2a} \tag{3}$$

Equation (1) yields the correct flight time for all possible elliptical paths between P_1 and P_2 for $0 \leq \theta < 2\pi$, which includes the cases for which the transfer angle θ is greater than or less than π and the cases for which the flight time t_F is greater than or less than the time t_m on the minimum-energy ellipse between P_1 and P_2, having semimajor axis $a_m = s/2$. The principal values α_0 and β_0 of the inverse sine functions used to solve Eqs. (2) and (3) are valid in Eq. (1) for the case $\theta \leq \pi$ and $t_F \leq t_m$. In the case that $\theta > \pi$, β in Eq. (1) is equal to $-\beta_0$, and in the case that $t_F > t_m$, α is equal to $2\pi - \alpha_0$. Clearly, $0 \leq \beta_0 \leq \alpha_0 \leq \pi$.

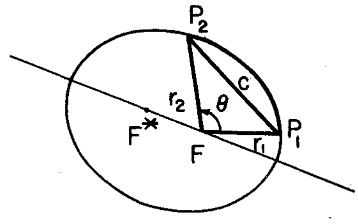

Fig. 1 Geometry of Lambert's problem.

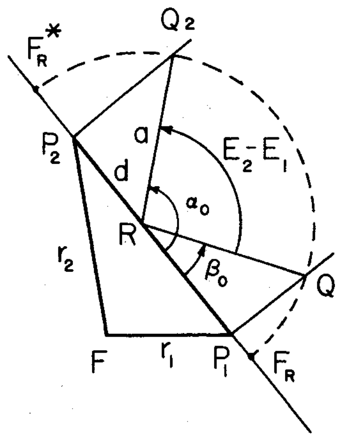

Fig. 2 Interpretation of the angles α_0 and β_0 for an elliptical orbit.

Geometrical Interpretation

The derivation of the geometrical interpretation of the angles α and β is based on two properties of elliptical motion: 1) the flight time satisfies Kepler's equation; and 2) the shape of the transfer orbit can be altered by moving the focus F and the vacant focus F^* without altering the flight time or the angles α and β as long as $r_1 + r_2$ and a remain unchanged in the process. This latter property is discussed in Ref. 2 as an artifice for simplifying the derivation of Eq. (1) for the flight time, and in Ref. 4 as a device for transforming the initial point into an apsidal point (see Fig. 3.5 of Ref. 2 or Fig. 6 of Ref. 4). Using this property, the focus and vacant focus can be moved to the locations F_R and F_R^* shown in Fig. 2, which define the rectilinear elliptical orbit between points P_1 and P_2, which has the same values of $r_1 + r_2$ and a and hence the same flight time, and α and β as the original orbit.

Kepler's equation for the flight time between two points in an elliptical orbit, whose locations are specified by the values of eccentric anomaly E is

$$\sqrt{\mu} t_F = a^{3/2} [E_2 - E_1 - e(\sin E_2 - \sin E_1)] \tag{4}$$

By comparing Eqs. (4) and (1), one can interpret the angles α and β as the values of eccentric anomaly on the rectilinear ellipse ($e = 1$) between P_1 and P_2, having the same values of a and $r_1 + r_2$.

The geometrical interpretation of α and β then follows the usual interpretation of eccentric anomaly. As shown in Fig. 2, one constructs an auxiliary circle of radius a centered at the center R of the rectilinear ellipse. Points Q_1 and Q_2 are the intersections of lines normal to the chord through points P_1 and P_2 with the auxiliary circle. The principal value angles α_0 and β_0 are the angles between the chord line and the auxiliary

Received Oct. 31, 1978; revision received April 24, 1979.

Index categories: Spacecraft Navigation, Guidance, and Flight Path Control; Lunar and Interplanetary Trajectories.

*Associate Professor, Department of Aeronautical and Astronautical Engineering. Associate Fellow AIAA.